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Quantum horizon fluctuations of an evaporating black hole

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Abstract

The quantum fluctuations of a black hole spacetime are studied within a low-energy effective field theory approach to quantum gravity. Our approach accounts for both intrinsic metric fluctuations and those induced by matter fields interacting with the gravitational field. Here we will concentrate on spherically symmetric fluctuations of the black hole horizon. Our results suggest that for a sufficiently massive evaporating black hole, fluctuations can accumulate over time and become significant well before reaching Planckian scales. In addition, we provide the sketch of a proof that the symmetrized two-point function of the stress-tensor operator smeared over a null hypersurface is actually divergent and discuss the implications for the analysis of horizon fluctuations. Finally, a natural way to probe quantum metric fluctuations near the horizon is briefly described.

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1. Introduction

In this paper, we study the quantum metric fluctuations near a black hole horizon within a low-energy effective field theory approach to quantum gravity. Although a fully satisfactory theory of quantum gravity is not available yet, it is expected that this approach will provide reliable information on phenomena involving length scales much larger than the Planck scale, rather independently of the microscopic details of the actual theory. The *stochastic gravity* formalism provides a useful framework for studying metric fluctuations in this context, especially for strong gravitational field situations, such as black hole background spacetimes (with masses much larger than the Planck mass) or in early cosmology (but well after the Planck time). In fact, it can be shown that the correlation functions for the metric fluctuations that one obtains are equivalent to those which would result from a entirely quantum field theoretic treatment (including the metric perturbations) to leading order in a $1/N$ expansion for a large number of matter fields N [1].

One starts by considering a *semiclassical gravity* solution [2, 3], where the spacetime geometry is described by a classical metric while the matter fields are quantized. The dynamics of the metric is governed by the semiclassical Einstein equation:

$$G_{ab}[g] = \kappa \langle \hat{T}_{ab}[g] \rangle_{\rm ren}, \tag{1}$$

where $\langle \hat{T}_{ab}[g] \rangle_{\rm ren}$ is the renormalized expectation value of the stress-tensor operator of the quantum matter fields and $\kappa = 8\pi / m_{\rm p}^2$ with $m_{\rm p}^2$ being the Planck mass. Both the semiclassical Einstein equation and the equation of motion for the matter fields evolving in that geometry, whose solution is needed to evaluate $\langle \hat{T}_{ab}[g] \rangle_{\rm ren}$, must be solved self-consistently.

The *stochastic gravity* formalism [4, 5] then provides a framework for studying the metric fluctuations around a semiclassical gravity solution. Its centrepiece is the Einstein–Langevin equation

$$G^{(1)}_{ab}[g+h] = \kappa \big\langle \hat{T}^{(1)}_{ab}[g+h] \big\rangle_{\rm ren} + \kappa \xi_{ab}[g], \tag{2}$$

which governs the dynamics of the metric fluctuations around the background metric g_{ab}. The superindex (1) indicates that only terms linear in the metric perturbations should be considered, and ξ_{ab} is a Gaussian stochastic source with vanishing expectation value and correlation function[1] $\langle \xi_{ab}(x) \xi_{cd}(x') \rangle_\xi = (1/2) \langle \{ \hat{t}_{ab}(x), \hat{t}_{cd}(x') \} \rangle$ (with $\hat{t}_{ab} \equiv \hat{T}_{ab} - \langle \hat{T}_{ab} \rangle$), where the term on the right-hand side, which accounts for the stress-tensor fluctuations within this Gaussian approximation, is commonly known as the *noise kernel* and denoted by $N_{abcd}(x, x')$.

2. Spherically-symmetric fluctuations for an evaporating black hole

2.1. Mean evolution

For a general spherically-symmetric metric, there always exists a system of coordinates in which it takes the form

$$\mathrm{d}s^2 = -\mathrm{e}^{2\psi(v,r)} (1 - 2m(v,r)/r)\, \mathrm{d}v^2 + 2\mathrm{e}^{\psi(v,r)}\, \mathrm{d}v\, \mathrm{d}r + r^2 (\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2). \tag{3}$$

This metric exhibits an apparent horizon at $r = 2M(v)$ when $M(v) = m(v, M(v))$. Assuming a state invariant under rotations for the quantum matter fields, the various components of the semiclassical Einstein equation associated with the metric in (3) become

$$\frac{\partial m}{\partial v} = 4\pi r^2 T^r_v, \tag{4}$$

$$\frac{\partial m}{\partial r} = -4\pi r^2 T^v_v, \tag{5}$$

$$\frac{\partial \psi}{\partial r} = 4\pi r T_{rr}, \tag{6}$$

where from now on we will simply use $T_{\mu\nu}$ to denote the expectation value $\langle \hat{T}_{\mu\nu}[g] \rangle_{\rm ren}$ and employ Planckian units (with $m_{\rm p}^2 = 1$). Furthermore, one can use the v component of the stress–energy conservation equation

$$\frac{\partial \left(r^2 T^r_v \right)}{\partial r} + r^2 \frac{\partial T^v_v}{\partial v} = 0, \tag{7}$$

to relate the T^r_v components on the horizon and moderately far from it (say, at $r \approx 6M$) since the second term in (7) can be neglected in the adiabatic regime (when $M \gg m_{\rm p}$). Moreover, in that case one can essentially recover Hawking's result for the radiation far from the horizon

[1] Throughout the paper we use the notation $\langle \cdots \rangle_\xi$ for stochastic averages over all possible realizations of the source ξ_{ab} to distinguish them from quantum averages, which are denoted by $\langle \cdots \rangle$.

replacing the constant mass by a slowly-varying time-dependent mass parameter. Taking into account this connection between energy fluxes and evaluating (4) on the apparent horizon, we finally get the equation governing the evolution of its size [6, 7]:

$$\frac{\mathrm{d}M}{\mathrm{d}v} = -\frac{B}{M^2}, \tag{8}$$

where B is a dimensionless parameter that depends on the number of massless fields and their spins and accounts for their corresponding grey-body factors.

Unless $M(v)$ is constant, the event horizon and the apparent horizon do not coincide. However, in the adiabatic regime their radii are related, differing by a quantity of higher order in $L_{\mathrm{H}} = m_{\mathrm{p}}/M$: $r_{\mathrm{EH}}(v) = r_{\mathrm{AH}}(v)(1 + O(L_{\mathrm{H}}))$.

2.2. *Induced fluctuations*

Following [8] we will concentrate here on spherically-symmetric fluctuations. However, it should be emphasized that this goes well-beyond previous studies based on effectively two-dimensional models, where dimensional reduction of the matter fields is performed at the very beginning. In contrast, here an infinite number of $l \neq 0$ modes rather than just the s-wave modes contribute to the $l = 0$ sector of the noise kernel.

The Einstein–Langevin equation for the spherically-symmetric sector of metric perturbations can be obtained by considering linear perturbations of $m(v, r)$ and $\psi(v, r)$, projecting the stochastic source that accounts for the stress-tensor fluctuations to the $l = 0$ sector, and adding it to the right-hand side of equations (4)–(6). We will focus our attention on the equation for the evolution of $\eta(v, r)$, the perturbation of $m(v, r)$:

$$\frac{\partial(m+\eta)}{\partial v} = -\frac{B}{(m+\eta)^2} + 4\pi r^2 \xi_v^r + O\left(L_{\mathrm{H}}^2\right), \tag{9}$$

where m is a semiclassical solution and ξ_v^r has been averaged over the whole solid angle.

Since the generation of Hawking radiation is especially sensitive to what happens near the horizon, from now on we will focus on the metric perturbations near the horizon and consider $\eta_{\mathrm{h}}(v) = \eta(v, 2M(v))$. Assuming that the fluctuations of the energy flux crossing the horizon and those far from it are exactly correlated, from (9) we have

$$\frac{\mathrm{d}\eta_{\mathrm{h}}(v)}{\mathrm{d}v} = \frac{2B}{M^3(v)}\eta_{\mathrm{h}}(v) + \xi(v), \tag{10}$$

where $\xi(v) \equiv \left(4\pi r^2 \xi_v^r\right)(v, r \approx 6M(v))$.

The correlation function for the spherically-symmetric fluctuation $\xi(v)$ is determined by the integral over the whole solid angle of the N_{vv}^{rr} component of the noise kernel, which is given by $(1/2)\left\langle\left\{\hat{t}_v^r(x), \hat{t}_v^r(x')\right\}\right\rangle$. The $l = 0$ fluctuations of the energy flux of Hawking radiation far from the horizon, characterized by $(1/2)\left\langle\left\{\hat{t}_v^r(x), \hat{t}_v^r(x')\right\}\right\rangle$, have been studied in [9] for a a black hole formed by gravitational collapse. Its main features are a correlation time of order M and a characteristic fluctuation amplitude of order ϵ_0/M^4 (this is the result of smearing the stress-tensor two-point function, which diverges in the coincidence limit, over a period of time of the order of the correlation time). The order of magnitude of ϵ_0 has been estimated to lie between $0.1B$ and B [9, 10]. For simplicity, we will consider quantities smeared over a time of order M. We can then introduce the Markovian approximation $\langle\xi(v)\xi(v')\rangle_\xi = (\epsilon_0/M^3(v))\delta(v - v')$, which coarse-grains the information on features corresponding to time-scales shorter than the correlation time M.

Solving equation (10) to express η_{h} in terms of ξ, one can obtain the amplitude of the fluctuations of η_{h}. Before doing that, it is convenient to change from the v coordinate to the mass function $M(v)$ for the background solution. The result is

$$\left\langle \eta_{\mathrm{h}}^2(M) \right\rangle_\xi = \left\langle \eta_{\mathrm{h}}^2(M_0) \right\rangle_\xi \left(\frac{M_0}{M}\right)^4 + \frac{\epsilon_0}{4B}\left[\left(\frac{M_0}{M}\right)^4 - 1\right]. \tag{11}$$

Provided that the fluctuations at the initial time corresponding to $M = M_0$ are negligible (much smaller than $\sqrt{\epsilon_0/4B} \sim 1$), the fluctuations become comparable to the background solution when $M \sim M_0^{2/3}$. For a black hole with an initial mass much larger than the Planck mass, fluctuations will become important well before reaching the Planckian regime. This result is in agreement with that previously obtained in [10].

3. Noise kernel near the horizon

When deriving (10) in last section, we assumed the existence of an exact correlation between the fluctuations of the ingoing energy flux crossing the horizon and the outgoing energy flux fluctuations far from it. Although this had been assumed in previous studies (see, however, [11] for an effectively two-dimensional model), a more careful analysis is needed since the typical variation time scale for ξ_v^r is M rather than M^3 and neglecting the last term in (7) is no longer justified for the fluctuations. In fact, as we will explain below, one can explicitly proof that such a direct correlation cannot exist.

Given the absence of a simple connection between the energy flux fluctuations near the horizon and far from it, an explicit calculation of the noise kernel near the horizon is, therefore, required. Even though a satisfactory approximation for the noise kernel in a black hole spacetime is not available yet, there are certain qualitative features that can be inferred in a fairly rigorous way. This follows from the fact that the Wightman function $G^+(x, x')$ is divergent in the limit of vanishing geodetic interval, i.e. when the length of the geodesic connecting x and x' goes to zero, and in particular for pairs of points connected by null geodesics. The noise kernel, which can be expressed as the product of two Wightman functions with a number of linear differential operators acting on them, exhibits a similar divergent behaviour. In those cases a finite result can be obtained by integrating the noise kernel with some appropriate smearing function. Ultimately the details of the smearing function will of course depend on the particular physical information one is trying to extract, and one expects that well-defined observable quantities related to the noise kernel will directly or effectively involve some kind of smearing that renders them finite.

The leading behaviour of the smeared noise kernel when the size of the smearing function along certain directions becomes small can be obtained for general Gaussian states in curved spacetime. The technical details will appear in [12], but the key steps can be summarized as follows. First, one does the calculation for the Minkowski vacuum in flat space, where the subtle products of distributions involved can be easily dealt with by working in Fourier space. In that case, the result for the noise kernel smeared around a null geodesic corresponding to a constant value of $u = t - x$ exhibits a leading contribution of order $1/\sigma_u \sigma_v^5 \sigma_r^2$ for the N_{vvvv} component (fluctuations of the energy flux crossing a hypersurface of constant u) in the limit of small σ_u, where σ_v is the size of the smearing along the null geodesic, σ_u is the smearing size along the 'transverse' null direction (corresponding to rays propagating in the opposite direction), and σ_r is the smearing size along the two orthogonal spatial directions.

This result can then be generalized to arbitrary Gaussian Hadamard states[2] in curved spacetime. For Hadamard states the expansion of the Wightman function for small geodetic

[2] It is usually assumed that physically acceptable states are of the Hadamard type, which gives rise to a regular expectation value of the stress-tensor operator.

interval $\sigma(x, x')$ is of the following form (see [2] for details):

$$G^+(x, x') = \frac{u(x, x')}{\sigma_+(x, x')} + v(x, x') \ln \sigma_+(x, x') + w(x, x'), \tag{12}$$

where $\sigma_+(x, x')$ is the geodetic interval (one half of the geodesic distance) for the geodesic connecting the pair of points x and x', with an additional small imaginary component added to the timelike coordinates. It is particularly convenient to make use of Riemann normal coordinates $\{y^\mu\}$, for which $\sigma_+(y, y') = (1/2)[-(y^0 - y'^0 - \mathrm{i}\varepsilon)^2 + (\vec{y} - \vec{y}')^2]$, since one can then define a Fourier transform in a fairly natural way. In those coordinates the most divergent contribution to $G^+(y, y')$, due to the first term on the right-hand side of (12), has the same form as for the Minkowski vacuum in flat space when using inertial coordinates. Furthermore, one can show that all the other contributions to the noise kernel, which come from products involving at least one of the other terms in (12), are subleading in the limit of small σ_u. Therefore, the leading contribution is the same as for the Minkowski vacuum, which is of order $1/\sigma_u \sigma_v^5 \sigma_r^2$, as we mentioned above.

From the previous result one can conclude that smearing over the horizon hypersurface is not sufficient to get a finite result: some additional smearing along the transverse (infalling) null direction, i.e. $\sigma_u \neq 0$, is also required. This seems to imply that once fluctuations are included it only makes sense to consider a horizon with a certain width. However, it is not clear how to characterize it in a precise and unambiguous manner. Fortunately there is a natural way to probe quantum metric fluctuations near the horizon and extract unambiguous and physically meaningful information by studying how Hawking radiation of a test field is modified when including the radiative corrections due to the metric fluctuations. A first step in that direction is to consider the propagation in the fluctuating metric of the set of null geodesics often employed in the derivation of the Hawking effect. A preliminary estimate suggests a substantial alteration of Hawking's original result. Nevertheless, we believe that this is probably due to a breakdown of the geometrical optics approximation and a full quantum field theoretic treatment may be required.

In addition, the results for the various limits of noise kernel smearings provide a way of proving that no direct correlation between the energy flux fluctuations crossing the horizon and far from it can exist. This follows from the fact that noise kernel smearings around a timelike curve are finite in the limit of vanishing smearing size along the three orthogonal spatial directions as long as the size of the smearing along the timelike direction does not vanish [12, 13]. In the adiabatic regime, (7) implies that the expectation value of the stress-tensor component T_v^r is independent of r for moderate values of r because the second term on the right-hand side of the equation can be neglected. If the same were true for the stochastic source ξ_v^r, one could directly relate the noise kernel on the horizon ($r = 2M$), where ξ_v^r and ξ_{vv} coincide, and far from it for each value of v. However, since curves of constant radius and angular coordinates are null at $r = 2M$ but timelike for larger radii, that would lead to a contradiction. A smearing of the noise kernel just along the v direction would be exactly related in both cases, but that cannot be true since it is finite in one case and divergent in the other.

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